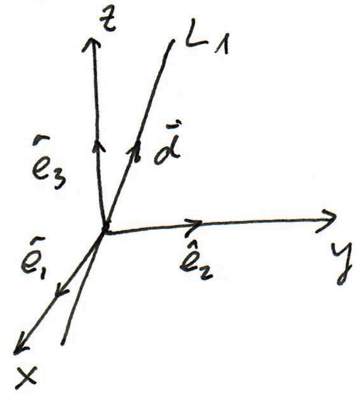
3.5 Concept of Subspace.

Consider the following sets in $\mathbb{R}^3$.

Lines through the origin.

a)



$\hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

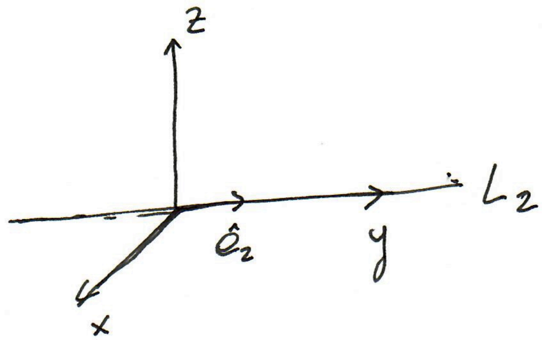
$\hat{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\hat{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\hat{d} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

or

b)



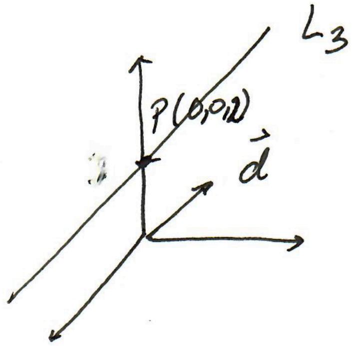
L_1 can be described by the vector equation:

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\vec{d} = t\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad t \text{ in } \mathbb{R}.$$

L_2 can be described by

$$\vec{x} = t\hat{e}_2, \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad t \in \mathbb{R}.$$

c)



$$\vec{d} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{p} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$L_3: \quad \vec{x} = t\vec{d} + \vec{p} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

Properties of lines L_1 and L_2

a) $\vec{0}$ is in L_1 and L_2, but $\vec{0}$ is not in L_3 (why?).

b) If $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ are in L_1

then $\vec{u} + \vec{v}$ is also in L_1

Proof. $\vec{u} = t_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{v} = t_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Then, $\vec{u} + \vec{v} = (t_1 + t_2)\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = t_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ in L_1

Similar for L_2.

However, for L_3

If $t=1$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \vec{u}$

If $t=2$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = \vec{v}$

Then, $\vec{u}+\vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}2\\4\\4\end{bmatrix} = \begin{bmatrix}3\\6\\7\end{bmatrix}$ is not in L_3

otherwise,

$$\begin{bmatrix}3\\6\\7\end{bmatrix} = t\begin{bmatrix}1\\2\\1\end{bmatrix} + \begin{bmatrix}0\\0\\2\end{bmatrix}$$

or $\left.\begin{array}{r} t = 3 \\ 2t = 6 \end{array}\right\}$ (1) (2)
$\boxed{t+2 = 7}$, (3)

Obviously, "t" is forced to be $t=3$ by equ. (1) Then, equ. (2) is also satisfied. But, equ. (3) is not satisfied, because

If $t=3$, $t+2=5 \neq 7$

c) Also, if $\vec{u}$ is in L_1 or L_2, then

$c\vec{u}$ is also in L_1 or L_2, for any c in $\mathbb{R}$.

Since if $\vec{u}$ is in L_1

then, there is t^* in $\mathbb{R}$ such that

$$\vec{u} = t^*\begin{bmatrix}1\\2\\1\end{bmatrix}$$

then,

$$c\vec{u} = ct^*\begin{bmatrix}1\\2\\1\end{bmatrix} = s\begin{bmatrix}1\\2\\1\end{bmatrix}. \checkmark \quad s \text{ in } \mathbb{R}.$$

Show that $2\begin{bmatrix}1\\2\\3\end{bmatrix}$ is not in L_3.

3.5 Subspaces, bases, dimension, and rank.

To motivate concept of Subspace.

Exercise Show that

$$S = \text{Span}\left(\overset{\vec{w}_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}}, \overset{\vec{w}_2}{\begin{bmatrix}-1\\2\\1\end{bmatrix}} \right)$$ is a subspace of $\mathbb{R}^3$.

we need to satisfy the three properties:

1) $\vec{0}$ is in S.
2) If $\vec{u}, \vec{v}$ are in S, then $\vec{u}+\vec{v}$ is in S.
3) If $\vec{u}$ is in S and c is a scalar, then $c\vec{u}$ is in S

Notice that

$$\vec{u} = \begin{bmatrix}0\\3\\2\end{bmatrix}, \quad \vec{v} = \begin{bmatrix}2\\5\\4\end{bmatrix} \text{ are in } S.$$

why? $\vec{u} = \vec{w}_1 + \vec{w}_2$, $\vec{v} = 3\vec{w}_1 + \vec{w}_2$

I) Also, $\vec{u}+\vec{v} = \begin{bmatrix}0\\3\\2\end{bmatrix} + \begin{bmatrix}2\\5\\4\end{bmatrix} = \begin{bmatrix}2\\8\\6\end{bmatrix}$ is in S

because

$$\vec{u}+\vec{v} = \begin{bmatrix}2\\8\\6\end{bmatrix} = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\2\\1\end{bmatrix} = 4\begin{bmatrix}1\\1\\1\end{bmatrix} + 2\begin{bmatrix}-1\\2\\1\end{bmatrix} \text{ is in } S.$$

$$\begin{aligned} c_1 - c_2 &= 2 \Rightarrow c_1 = 2 + c_2 \\ c_1 + 2c_2 &= 8 \\ c_1 + c_2 &= 6 \Rightarrow 2 + c_2 + c_2 = 6 \Rightarrow 2 + 2c_2 = 6 \end{aligned}$$

$$c_2 = \frac{4}{2} = 2$$

$$\Rightarrow c_1 = 2 + 2 = 4$$

II) $\vec{0}$ is in S

because $0\begin{bmatrix}1\\1\\1\end{bmatrix}=\vec{0}$ is in S.

III) For any $\vec{u}$ in S

$$\vec{u}=c_1\begin{bmatrix}1\\1\\1\end{bmatrix}+c_2\begin{bmatrix}-1\\2\\1\end{bmatrix}$$

If d is a scalar

$$d\vec{u}=dc_1\begin{bmatrix}1\\1\\1\end{bmatrix}+dc_2\begin{bmatrix}-1\\2\\1\end{bmatrix}$$

$$=f_1\begin{bmatrix}1\\1\\1\end{bmatrix}+f_2\begin{bmatrix}-1\\2\\1\end{bmatrix}$$ is in S.

<u>Definition</u> A set $S \subset \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if

a) $\vec{0}$ is in S

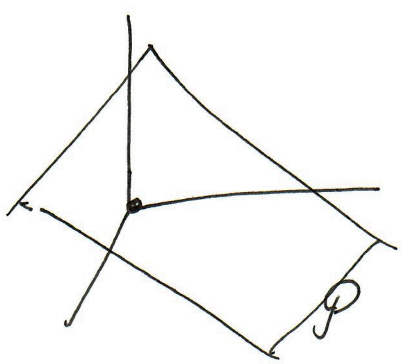
Closed under addition b) If $\vec{u}, \vec{v}$ in S, then $\vec{u}+\vec{v}$ is in S.

Closed under scalar multiplication c) If $\vec{u}$ is in S and c is in $\mathbb{R}$ then, $c\vec{u}$ is in S.

<u>Example</u> Any plane through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

In fact,

$$\mathcal{P}: \ c_1\vec{u} + c_2\vec{v} = c_1\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + c_2\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$



Clearly,

a) $\vec{0}$ is in $\mathcal{P}$
Just choose $c_1 = 0$, $c_2 = 0$.

b) If u_1 and u_2 are in $\mathcal{P}$.

then,
$$\left.\begin{array}{l} \vec{u}_1 = c_1\vec{u} + c_2\vec{v} \\ \vec{u}_2 = d_1\vec{u} + d_2\vec{v} \end{array}\right\} \Rightarrow \vec{u}_1 + \vec{u}_2 = (c_1+d_1)\vec{u} + (d_1+d_2)\vec{v}$$
$$= f_1 u + f_2\vec{v}$$

where $f_1 = c_1 + d_1$, $f_2 = c_2 + d_2$.

3.5 #8) $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ in } \mathbb{R}^3, \text{ such that } |x-y| = |y-z| \right\}$

$\vec{0}$ is in S.

Let's consider two vectors in S

$$\vec{u}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$u_1 + u_2 = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} \qquad |4-2| = 2 = |2-0|$$

If $\vec{u}_2 = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$

$$\vec{u}_1 + \vec{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

therefore, $\vec{u}_1 + \vec{u}_2$ is not in S.

Is $c\vec{u}$ in S if $\vec{u}$ is in S, for any c?

because, $\underset{|x-y|}{|3-12|} = 9 \neq 11 = \underset{|y-z|}{|12-1|}$

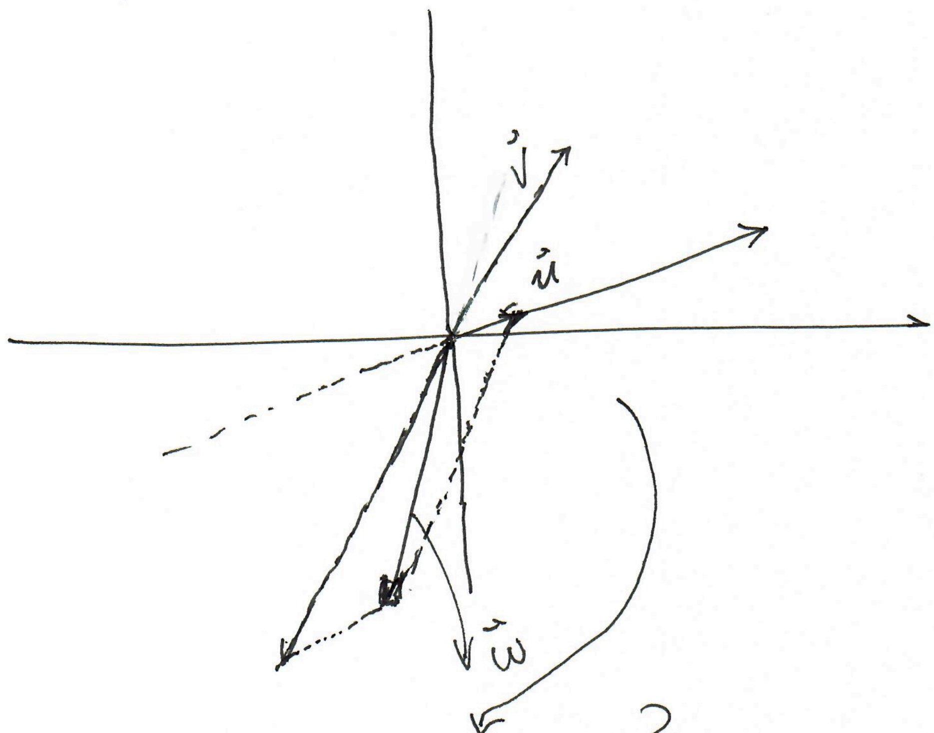
Therefore, S is not a subspace.

#6) $z = 2x,\ y = 0$ is a subspace?

$\vec{u} = \begin{bmatrix} x \\ 0 \\ 2x \end{bmatrix}$ represent all vector in S for arb. x, then if $\vec{v}$ is in S, $\vec{v} = \begin{bmatrix} z \\ 0 \\ 2z \end{bmatrix}$ and $\vec{u} + \vec{v} = \begin{bmatrix} x+z \\ 0 \\ 2(x+z) \end{bmatrix}$

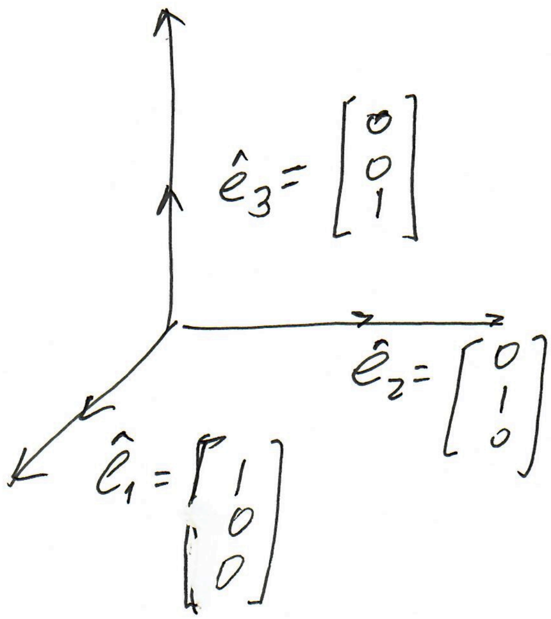
$\vec{0}$ is in S b/c ~~if~~ $x=0$ $\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, also $c\vec{u}$ is in S, b/c $c\vec{u} = \begin{bmatrix} cx \\ 0 \\ 2cx \end{bmatrix}$

Any two vectors <u>not in the same line</u> span $\mathbb{R}^2$.



So $\operatorname{span}(\vec{u},\vec{v}) = \mathbb{R}^2$ is a Subspace.

In $\mathbb{R}^3$



Any Vector $\hat{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in $\mathbb{R}^3$

Can be written as

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} =$$

$$= b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3$$

(1) Then $\operatorname{span}(\hat{e}_1, \hat{e}_2, \hat{e}_3) = \mathbb{R}^3$

(2) Also, $B_3 = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ is lin. indep.

we say that B_3 is the "standard basis" for $\mathbb{R}^3$.

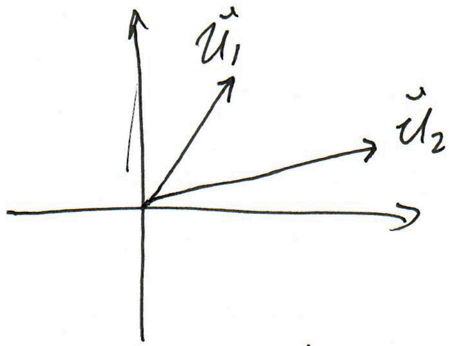
Def If S is a Subspace of $\mathbb{R}^n$
a Set of Vectors $\mathcal{B}$ in S is a basis for S if

① S is linearly independent.

② $\text{Span}(\mathcal{B}) = S$.

Other bases in $\mathbb{R}^2$

Consider
$$\mathcal{B} = \left\{ \overset{\vec{u}_1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}, \overset{\vec{u}_2}{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} \right\}$$



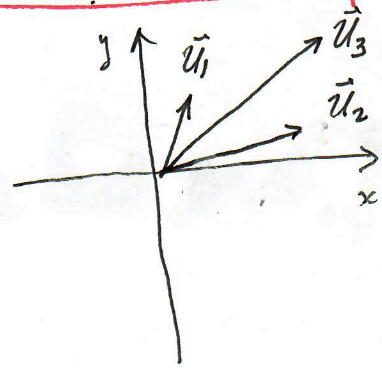
① why lin. indep.?

② why $\text{Span}(\mathcal{B}) = \mathbb{R}^2$?

How many bases are there in $\mathbb{R}^2$?

– $\boxed{\text{Extends this discussion to } \mathbb{R}^3 \longrightarrow \mathbb{R}^n.}$

Is the set
$$H = \left\{ \overset{\vec{u}_1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}, \overset{\vec{u}_2}{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}, \overset{\vec{u}_3}{\begin{bmatrix} 5 \\ 5 \end{bmatrix}} \right\}$$
a basis for $\mathbb{R}^2$



Consider the set

$$H = \left\{ \begin{array}{c} \vec{v}_1 \\ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \end{array}, \begin{array}{c} \vec{v}_2 \\ \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \end{array}, \begin{array}{c} \vec{v}_3 \\ \begin{bmatrix} -1 \\ -4 \\ -7 \end{bmatrix} \end{array} \right\}$$

Is this set a basis for $\mathbb{R}^3$?

Determine if they are linearly indep. or not?

Consider

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

and find the coeffs c_1, c_2 and c_3.

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 5 & -4 & 0 \\ -3 & -1 & -7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 5 & -10 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \boxed{c_2 = 2c_3}$$

$$c_1 = -2c_2 + c_3 = -4c_3 + c_3$$

$$\boxed{c_1 = -3c_3}, \quad \boxed{c_3 \text{ free}}$$

Then

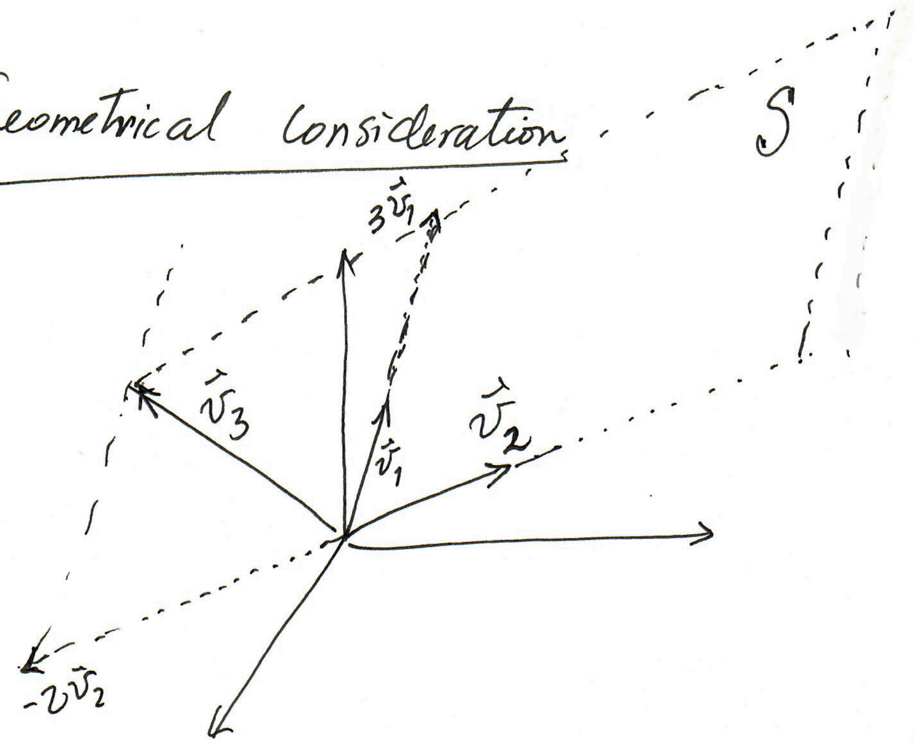
$$(-3c_3)\vec{v}_1 + (2c_3)\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

If $c_3 \neq 0$ $\quad -3\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3 = \vec{0}$

or $\quad \boxed{\vec{v}_3 = 3\vec{v}_1 - 2\vec{v}_2}$ So $\vec{v}_3$ is linearly dep. on $\vec{v}_1$ and $\vec{v}_2$.

As a consequence, the set of vectors in H is not a basis for $\mathbb{R}^3$

Geometrical consideration



$\vec{v}_3$ is in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$ = Plane through origin. $= S$

What is $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

All possible linear comb.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = c_1\vec{v}_1 + c_2\vec{v}_2$$

$$+ c_3(3\vec{v}_1 - 2\vec{v}_2) = c_1\vec{v}_1 + c_2\vec{v}_2$$

$$= c_1\vec{v}_1 + c_2\vec{v}_2 - 3c_3\vec{v}_1 - 2c_3\vec{v}_2$$

$$= (c_1 - 3c_3)\vec{v}_1 + (c_2 - 2c_3)\vec{v}_2$$

$$= d_1\vec{v}_1 + d_2\vec{v}_2$$

So $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ = Same plane in $\mathbb{R}^3$ $= S$

So the Set

① $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \right\}$ Spans Plane S.

② Also, B is a linearly indep. set. (why?)

Therefore, B is a basis for plane S.

Can you complete B to form a basis for $\mathbb{R}^3$.

- Discuss geometrically this idea.

3.5 b)

Thm 3.2.3

Any two bases for a Subspace S of $\mathbb{R}^n$ have the same number of Vectors.

Def. The number of vectors in a basis for S is called the dimension of S, and it's denoted as

$$\dim S.$$

Examples: $\dim(\mathbb{R}^2) = 2$, $\dim(\mathbb{R}^3) = 3$